

Name: _____ September 1, 2016

All unnecessary electronics must be turned off and out of sight. This means no calculator, cellular phones, iPods, wearing of headphones, or anything of the sort. This is a closed book and closed notes test.

Problem 1: (10 points) Find the counterclockwise circulation and the outward flux of the field $\vec{F} = (-\sin y)\hat{i} + (x \cos y)\hat{j}$ around and across the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$.

Solution: Let C be the boundary of the square and D be the square itself. Then the circulation is given by

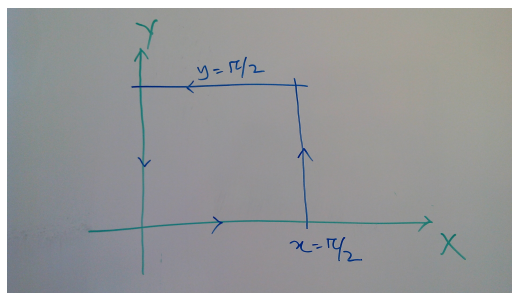


Figure 1: The square.

$$\begin{aligned}
 \text{Circulation} &= \oint_C (-\sin y) dx + (x \cos y) dy \\
 &= \iint_D \left[\frac{\partial}{\partial x}(x \cos y) - \frac{\partial}{\partial y}(-\sin y) \right] dxdy \quad (\text{by Green's theorem, 3 points}) \\
 &= \int_0^{\pi/2} \int_0^{\pi/2} 2 \cos y \, dxdy \\
 &= 2 \frac{\pi}{2} \int_0^{\pi/2} \cos y \, dy \\
 &= \pi \sin y \Big|_0^{\pi/2} = \pi \quad (2 \text{ points}).
 \end{aligned}$$

The outward flux over the square is

$$\begin{aligned}
 \text{Flux} &= \int_0^{\pi/2} \int_0^{\pi/2} (-\sin y) dy - (x \cos y) dx \\
 &= \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{\partial}{\partial x}(-\sin y) + \frac{\partial}{\partial y}(x \cos y) \right] dxdy \quad (\text{by Green's theorem, 3 points}) \\
 &= - \int_0^{\pi/2} \int_0^{\pi/2} x \sin y \, dxdy \\
 &= - \int_0^{\pi/2} \frac{1}{2} \cdot \frac{\pi^2}{4} \sin y \, dy \\
 &= \frac{\pi^2}{8} \cos y \Big|_0^{\pi/2} = -\frac{\pi^2}{8} \quad (2 \text{ points}).
 \end{aligned}$$

Problem 2: (10 points) Find the surface area of the portion of the cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$.

Solution: Parametrize the cylinder by

$$\vec{r}(u, v) = \sqrt{10} \cos u \hat{i} + \sqrt{10} \sin u \hat{j} + v \hat{k}, \quad 0 \leq u \leq 2\pi, \quad -1 \leq v \leq 1 \quad (3 \text{ points})$$

The normal vector to the surface is given by

$$\begin{aligned} \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sqrt{10} \sin u & \sqrt{10} \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \sqrt{10} \cos u \hat{i} + \sqrt{10} \sin u \hat{j} \quad (3 \text{ points}). \end{aligned}$$

Therefore

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = \sqrt{10} \quad (1 \text{ point}).$$

Consequently, the surface area of the cylinder is given by

$$\begin{aligned} &\int \int \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv \quad (2 \text{ points}) \\ &= \int_{-1}^1 \int_0^{2\pi} \sqrt{10} du dv \\ &= 4\sqrt{10}\pi \quad (1 \text{ point}). \end{aligned}$$

ALTERNATIVE METHOD

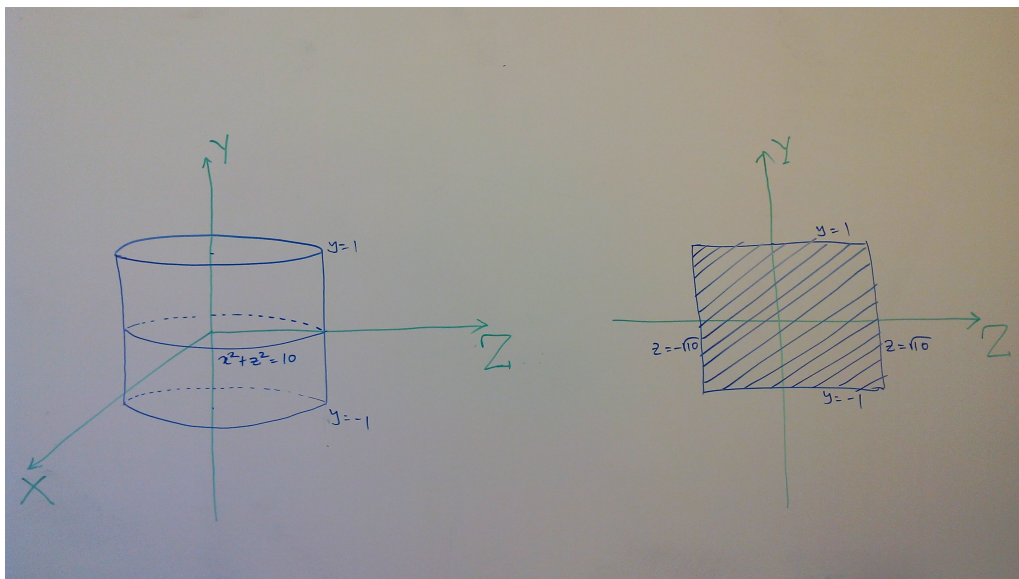


Figure 2: Left: The cylinder itself, Right: Shadow of the cylinder on the yz -plane.

The equation of the cylinder can be written as $f(x, y, z) = 0$, where $f(x, y, z) = x^2 + z^2 - 10$. The gradient of f is

$$\vec{\nabla} f = 2x\hat{i} + 0\hat{j} + 2z\hat{k} \quad (2 \text{ points}).$$

The base of the cylinder is on the xz -plane. Therefore the shadow of the cylinder on the xz -plane is just the perimeter of the circle $x^2 + z^2 = 10$. So we will take the shadow on the yz -plane (You may also take the shadow on the xy -plane). Shadow of this cylinder on the yz -plane is the rectangle which is bounded by the lines $y = \pm 1$ and $z = \pm\sqrt{10}$. We also notice that the two sides ($x = \pm\sqrt{10 - z^2}$) of the cylinder have the same surface area. But after taking the shadow, they overlap. Therefore the area of the cylinder is given by

$$\begin{aligned} & 2 \int_{-\sqrt{10}}^{\sqrt{10}} \int_{-1}^1 \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \hat{i}|} dydz \quad (3 \text{ points}) \\ = & 2 \int_{-\sqrt{10}}^{\sqrt{10}} \int_{-1}^1 \frac{2\sqrt{x^2 + z^2}}{2x} dydz \quad (2 \text{ points}) \\ = & 2 \int_{-\sqrt{10}}^{\sqrt{10}} \int_{-1}^1 \frac{\sqrt{10}}{\sqrt{10 - z^2}} dydz \quad (\text{because } x^2 + z^2 = 10) \\ = & 2\sqrt{10} \int_{-\sqrt{10}}^{\sqrt{10}} \frac{2}{\sqrt{10 - z^2}} dz \\ = & 4\sqrt{10} \sin^{-1} \left(\frac{z}{\sqrt{10}} \right) \Big|_{-\sqrt{10}}^{\sqrt{10}} \\ = & 4\pi\sqrt{10} \quad (3 \text{ points}). \end{aligned}$$